

# 3 Combinatorial Logic

## Student Group

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## 3. Combinatorial Logic

### introductory example

Fig. 1: Simulation of a 7-segment encoder and display

The combinatorial logic shown in `<impref pic1>` enables to output distinct logic values for each logic input. When you change the input nibble you can see that the correct number appears on the 7-segment-display. By clicking onto the bits of the input nibble, you can change the number.

Tasks:

1. Which output  $Y_0$  ...  $Y_6$  is generated from the input nibble 1000? Which from 1001?
2. Is the output only depending on the input? Is there a dependence on the history?

### 3.1 Combinatorial Circuit

Up to now, we looked onto simple logic circuits. These are relatively easy to analyze and synthesize (=develop). The main question in this chapter is: how can we set up and optimize logic circuits?

In the following we have a look onto combinatorial circuits. These are generally logic circuits with

- $n$  inputs  $X_0, X_1, \dots, X_{n-1}$
- $m$  outputs  $Y_0, Y_1, \dots, Y_{m-1}$
- no "memory", that is: a given set of input bits results in a distinct output

They can be description by

- truth table
- boolean formula
- hardware description language

The ladder one is not in the focus of this course.

The applications range:

- (simple) half/full adder
- [digital comparators](#) (logic circuit to compare 2 values)
- Multiplexer / demultiplexer
- Arithmetic logic units in microcontrollers and processors
- much more

#### 3.1.1 Example

In order to understand the synthesis of a combinatoric logic we will follow a step-by-step example for

this chapter.

Imagine you are working for a company called “mechatronics and robotics”. One customer wants to have an intelligent switch as input device connected to a microcontroller for controlling an oven. For this project “Therm-o-Safety” he needs a combinatoric logic:

- The intelligent switch has 4 user selectable positions: \$1\$, \$2\$, \$3\$, \$4\$
- Additionally there are 2 non-selectable positions for the case of failure.
- The output \$Y=1\$ will activate a temperature monitoring.
- The temperature monitoring has to be active for \$3\$ and \$4\$ and in case of a major failure. A major failure is for example, when the switch position is unclear. In this case the input of the combinatorial circuit is “ON”.
- There are no other cases of inputs.

This requirements are put into a truth table:

Therm-o-Safety				
Input	X2	X1	X0	Y
	0	0	0	-
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	1
	1	0	1	-
OFF	1	1	0	0
ON	1	1	1	1



Fig. 2: Therm-o-Safety truth table

figure 2 shows one implementation of this requirements. The inputs 001 ... 011 represent the inputs \$1\$...\$4\$. The cases of failure are coded with 110 and 111. The output \$Y\$ is activated as requested. For the two combinations 000 and 101 there is no output value defined. Depending on the requirements for a project these shall either better be 0 or 1 or the output of these does not matter. We had this “does not matter” before: the technical term is “I don't care”, and it is written as a - or a x.

By this, we have done the first step in order to synthesize the requested logic.

### 3.1.2 Sum of Products

Now, we want to investigate some of the input combinations (= lines in the truth table). At first, we have a look onto the input combination 011, where the output has to be \$Y=1\$.

If this input combination would be the only one for the output of \$Y=1\$, the following could be stated: “\$Y=1\$ (only) when the \$X\_0\$ is \$1\$ AND \$X\_1\$ is \$1\$ AND \$X\_2\$ is \$0\$”. It can also be re-arranged to:

“ $Y=1$  (only) when the  $X_0$  is  $1$  AND  $X_1$  is  $1$  AND  $X_2$  is not  $1$  ”.

This statement is similar to  $X_0 \cdot X_1 \cdot \overline{X_2}$ . The used conjunction results only in  $1$ , when all inputs are  $1$ . The negation of  $X_2$  takes account of the fact, that  $X_2$  has to be  $0$ .

Fig. 3: Therm-o-Safety truth table - first analysis

Therm-o-Safety				
Input	X2	X1	X0	Y
	0	0	0	-
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	1
	1	0	1	-
OFF	1	1	0	0
ON	1	1	1	1

$$\overline{X_2} \& X_1 \& X_0$$

figure 3 shows the boolean expression for this combination. In figure 4, this boolean expression is converted into a structure with logic gates.

Fig. 4: logic circuit for the combination '011'

With the same idea in mind, we can have a look for the other combinations resulting in  $Y=1$ . These are the combinations 100 and 111:

- For 100 The statement would be: “ $Y=1$  (only) when the  $X_0$  is  $0$  AND  $X_1$  is  $0$  AND  $X_2$  is  $1$ ”. Similarly to the combination above this leads to:  $\overline{X_0} \cdot \overline{X_1} \cdot X_2$ .
- For 111, the boolean expression is  $X_0 \cdot X_1 \cdot X_2$ .

**Note!**

- Each row in a truth table (=one distinct combination) can be represented by a **minterm** or **maxterm**
- A **minterm** is the conjunction (AND'ing) of all inputs, where under certain instances a negation have to be used
- In a minterm an input variable with 0 has to be negated, in order to use it as an input for the AND.  
e.g.  $X_0 = 0$  AND  $X_1 = 1 \quad \rightarrow \quad \overline{X_0} \cdot X_1$
- A minterm results in a output of 1

Fig. 5: Therm-o-Safety truth table - sum of products

Therm-o-Safety					
Input	X2	X1	X0	Y	minterm
	0	0	0	-	
1	0	0	1	0	
2	0	1	0	0	
3	0	1	1	1	$\overline{X_2} \cdot X_1 \cdot X_0$
4	1	0	0	1	$X_2 \cdot \overline{X_1} \cdot \overline{X_0}$
	1	0	1	-	
OFF	1	1	0	0	
ON	1	1	1	1	$X_2 \cdot X_1 \cdot X_0$

In [figure 5](#) all minterms for  $Y=1$  are shown. The [figure 6](#) depicts all the logic circuits for the three minterms. These lead to the outputs  $Y'$ ,  $Y''$ , and  $Y'''$ .

Fig. 6: logic circuit for the combinations '100', '110', '111'

For the final step we have to combine the single results for the minterms. The output has to be  $1$  when at least one of the minterms is  $1$ . Therefore, the minterms have to be connected disjunctive:

$$Y = Y' \quad + \quad Y'' \quad + \quad Y''' \quad || \quad Y = (X_0 \cdot X_1 \cdot \overline{X_2}) \quad + \quad (\overline{X_0} \cdot \overline{X_1} \cdot X_2) \quad + \quad (X_0 \cdot X_1 \cdot X_2)$$

This leads to the logic circuit shown in [figure 7](#). Here, you can input the different combinations by clicking onto the bits of the input nibble.

Fig. 7: logic circuit for therm-o-safety

### Note!

- The disjunction of the minterms is called **sum of products, SoP, disjunctive normal form** or **DNF**.
- When all inputs are used in each of the minterms the normal form is called **full disjunctive normal form**
- When synthesizing a logic circuit by sum of products, all 'don't care' terms outputting  $0$ .

We have seen, that the sum of products is one tool to derive a logic circuit based on a truth table. Alternatively it is also possible to insert an intermediate step, where the logic formula is simplified.

In the following one possible optimization is shown:

$$\begin{aligned} Y &= (X_0 \cdot X_1 \cdot \overline{X_2}) \quad + \quad (\overline{X_0} \cdot \overline{X_1} \cdot X_2) \quad + \quad (X_0 \cdot \overline{X_1} \cdot X_2) \quad + \quad (\overline{X_0} \cdot X_1 \cdot \overline{X_2}) \\ &= (\overline{X_0} \cdot \overline{X_1} \cdot X_2) \quad + \quad (X_0 \cdot \overline{X_1} \cdot X_2) \quad + \quad (\overline{X_0} \cdot X_1 \cdot \overline{X_2}) \quad + \quad (X_0 \cdot X_1 \cdot \overline{X_2}) \\ &= (\overline{X_0} \cdot \overline{X_1} \cdot X_2) \quad + \quad ((X_0 \cdot \overline{X_1}) \cdot X_2) \quad + \quad (\overline{X_0} \cdot X_1 \cdot \overline{X_2}) \quad + \quad ((X_0 \cdot X_1) \cdot \overline{X_2}) \\ &= (\overline{X_0} \cdot \overline{X_1} \cdot X_2) \quad + \quad ((X_0 \cdot \overline{X_1}) \cdot X_2) \quad + \quad ((X_0 \cdot X_1) \cdot \overline{X_2}) \quad + \quad ((\overline{X_0} \cdot X_1) \cdot \overline{X_2}) \\ &= (\overline{X_0} \cdot \overline{X_1} \cdot X_2) \quad + \quad ((X_0 \cdot \overline{X_1}) \cdot X_2) \quad + \quad ((X_0 \cdot X_1) \cdot \overline{X_2}) \quad + \quad ((\overline{X_0} \cdot X_1) \cdot \overline{X_2}) \end{aligned}$$

### 3.1.3 Product of Sums

In the sub-chapter before we had a look onto the combinations which generates an output of  $Y=1$  by means of the AND operator. Now we are investigating the combinations with  $Y=0$ . Therefore, we need an operator, which results in  $0$  for only one distinct combination.

The first combination to look for is  $001$ . If this input combination would be the only one for the output of  $Y=0$ , the following could be stated:

" $Y=0$  (only) when the  $X_0$  is  $1$  AND  $X_1$  is  $0$  AND  $X_2$  is  $0$ ".

With having the duality in mind (see cpt. [The Set of Rules](#)) the opposite is also true:

" $Y=1$  when  $X_0$  is  $0$  OR  $X_1$  is  $1$  OR  $X_2$  is  $1$ ".

This is the same like:  $\overline{X_0} + X_1 + X_2$

The booleand operator we need here is the OR-operator.

Similarly, the combinations  $010$  und  $110$  can be transformed. Keep in mind, that this time we are looking for a formula with results in  $0$  only for the given one distinct combination.

#### Note!

- A **maxterm** is the disjunction (OR'ing) of all inputs, where unter certain instances a negation have to be used.
- In a maxterm an input variable with  $1$  has to be negated, in order to use it as an input for the OR.
- A maxterm results in a output of  $0$

The [figure 8](#) shows all the maxterms for the Therm-o-Safety example.

Fig. 8: Therm-o-Safety truth table

Therm-o-Safety						
Input	X2	X1	X0	Y	minterm	maxterm
	0	0	0	-		
1	0	0	1	0		$X_2 + X_1 + \overline{X_0}$
2	0	1	0	0		$X_2 + \overline{X_1} + X_0$
3	0	1	1	1	$\overline{X_2} \cdot X_1 \cdot X_0$	
4	1	0	0	1	$X_2 \cdot \overline{X_1} \cdot \overline{X_0}$	
	1	0	1	-		
OFF	1	1	0	0		$\overline{X_2} + \overline{X_1} + X_0$
ON	1	1	1	1	$X_2 \cdot X_1 \cdot X_0$	

The formulas of figure 8 can again be transformed into gate circuits (figure 9). Here, only for the inputs '001', '010', '110' one of the outputs \$Y\$, \$Y''\$ or \$Y'''\$ is \$0\$.

Fig. 9: logic circuit for the combinations '001', '010', '110'

When these intermediate outputs \$Y\$, \$Y''\$, \$Y'''\$ are used as an input for an AND-gate the result in output will get \$0\$ when at least one of the intermediate outputs are \$0\$. This results in another way to synthesize the Therm-o-Safety (see figure 10)

Fig. 10: logic circuit for therm-o-safety

Also the products of sum can be simplified:

$$Y = (\overline{X_0} + X_1 + X_2) \cdot (\overline{X_0} + \overline{X_1} + X_2) \cdot (\overline{X_0} + \overline{X_1} + X_2) \cdot (\overline{X_0} + \overline{X_1})$$

This result \$Y\$ by the sum of products is different compared to the result in product of sums:

- product of sums:  $Y = (\overline{X_0} \cdot \overline{X_1} \cdot X_2) + (X_0 \cdot X_1)$
- sum of products:  $Y = (\overline{X_0} + X_1 + X_2) \cdot (\overline{X_0} + \overline{X_1})$

In this case these results cannot be transformed into each other with the means of boolean rules.

**Note!**

- The disjunction of the maxterms is called **products of sum, PoS, conjunctive normal form** or **CNF**.
- When all inputs are used in each of the minterms the normal form is called **full conjunctive normal form**

- When snytesizing a logic circuit by sum of products, all 'don't care' terms outputing \$1\$
- The products of sum is the DeMorgan dual of the sum of products **if** there are no don't care terms. Otherwise the results cannot be transformed into each other with the means of boolean rules.

## 3.2 Karnaugh Map

### 3.2.1 Introduction with example

For our therm-o-safety example we found two possible gate logics which can produce the required output. We have also seen, that optimizing the terms (i.e. the min- or maxterms) is often possible. But we do not know how we can find the optimum implementation.

For this, we try to interpret the inputs of our example as dimensions in a multidimensional space. The three input variables  $X_0$ ,  $X_1$ ,  $X_2$  span a 3-dimensional space. The point 000 is the origin of this space. The three combinations 001, 010, 100 are onto the  $X_0$ -,  $X_1$ -, and  $X_2$ -axis, respectively (see figure 12 (a)). The other combinations can be reached by adding these axis values together (see figure 12 (b)+(c)). This is similar to the situation of a two dimensional or three dimensional vector. Three inputs result in this representation in the edges of a cube.

In the following pictures of this representation the values are shown as:

- green dot, when the result is 1
- red dot, when the result is 0
- grey dot, when the result is don't care

In the figure 12 (d) the situation  $X_0=1$ ,  $X_1=1$ ,  $X_2=0$  is shown.

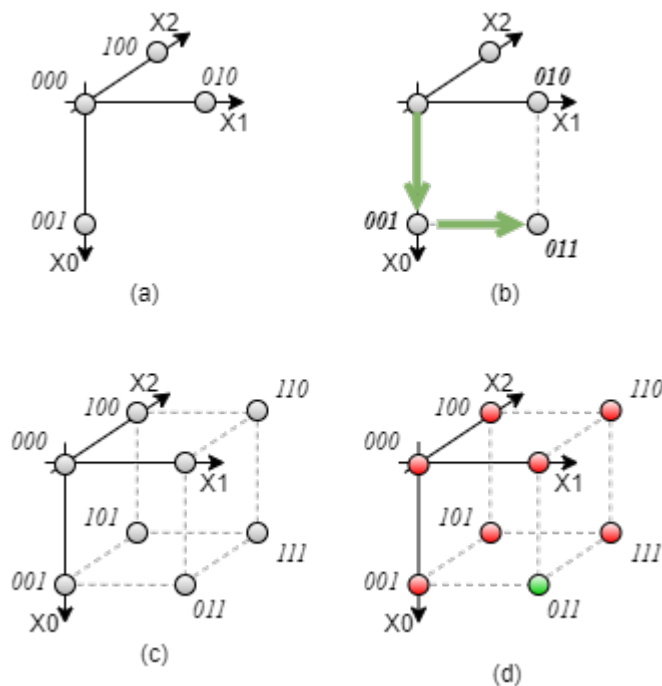


Fig. 12: 3 dimensional cube representation

There is also an alternative way to look onto this representation:

- The value 1 (independent from the inputs) lead to all positions are 1
- A single input equal 1 (independent from all other inputs, i.e. all others are don't care) lead in our example to the edges of a side surface of the cube.  
 In our example  $X_1=1$  lead to the situation shown in figure 13 (a). When investigating the shown green dots  $010, 011, 110, 111$  it is visible that the middle value (= the value for  $X_1$ ) is the same.  
 Generally: A single input equal 1 (independent from all other inputs) lead to a structure one dimension smaller than the number of inputs (In our example: 3 inputs  $\rightarrow$  two dimensional structure = surface).
- Multiple given inputs equal 1 lead to smaller structures correspondingly. In our example:  $X_0=1$  and  $X_1=1$  ( $=X_0 \cdot X_1$ ) result in the two edges on a corner of the cube (figure 13 (b)). For this coordinates  $(011, 111)$  the last two values are the same.
- A minterm (=1 as an output) in our example is given by the intersection of all surfaces for the individual dimensions. In our example:  $X_0=1$  and  $X_1=1$  and  $X_2=0$  ( $=X_0 \cdot X_1$ ) result in the two edges on a corner of the cube (figure 13 (b))

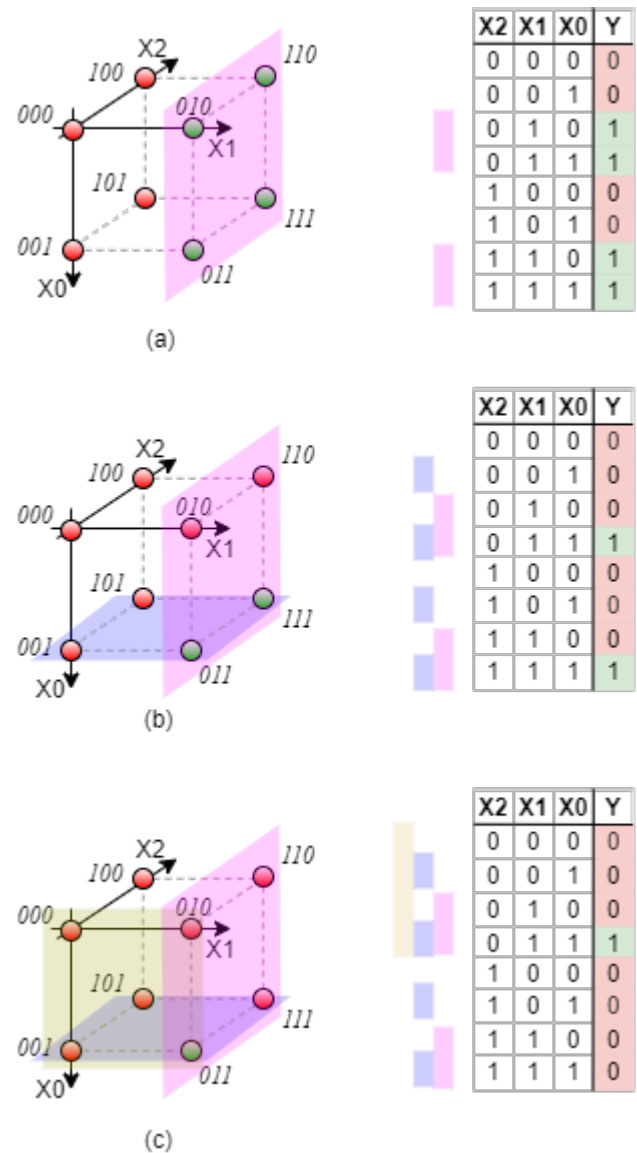


Fig. 13: examples in 3 dimensional cube representation

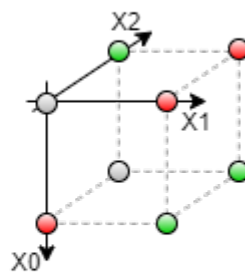
With this representation in mind, we can simplify other representations much more simpler. \# One

example for this would be to represent the formula:  $Y = X_0 \cdot X_1 \cdot \overline{X_2} + X_2 \cdot X_1 + X_0 \cdot X_1$ . By drawing this into the cube one will see that it represents only a side surface of the cube. It can be simplified into  $Y = X_1$ .

We can also try to interpret our Therm-o-Safety truth table. The figure 14 shows the corresponding cube. The problem here is, that it is a bit unhandy to reduce a three dimensional cube onto a flat monitor or a paper. It will also get more stressfull for higher dimensions.

Fig. 14: Therm-o-Safety in multi-dimensional space

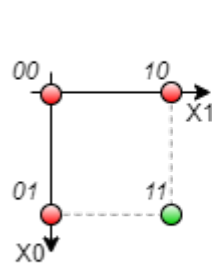
Therm-o-Safety				
Input	X2	X1	X0	Y
	0	0	0	-
1	0	0	1	0
2	0	1	0	0
3	0	1	1	1
4	1	0	0	1
	1	0	1	-
OFF	1	1	0	0
ON	1	1	1	1



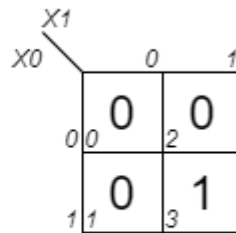
Therefore, we try to find a better way to sketch the coordinates, before we simplify our Therm-o-Safety. We take one step back and look onto a two dimensional and simple example:  $Y = X_0 \cdot X_1$ . This can be shown as the edges of a square (figure 11 (a)). We will in the future we will write this as in figure 11 (b). This diagram is also called **karnaugh map**.

$X_0 \cdot X_1$			
dec	X1	X0	Y
0	0	0	0
1	0	1	0
2	1	0	0
3	1	1	1

(a)



(b)



(c)

Fig. 11: two dimensional

[interactive example](#)

### Exercise 3.1.x Further Questions

1. compare the results with the output given [here](#) (the output  $y$  can be changed by clicking onto it)

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