5 Magnetic Circuits

Student Group

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Table of Contents

5. Magnetic Circuits

> For this and the following chapter the online Book 'DC Electrical Circuit Analysis - A Practical Approach' is strongly recommended as a reference. In detail this is chapter 10.3 Magnetic Circuits

In the previous chapters, we got accustomed to the magnetic field. During this path, some similarities from the magnetic field to the electric circuit appeared (see figure 1).

Fig. 1: Similarities magnetic Circuit vs electric Circuit

In this chapter, we will investigate, how far we come with such an analogy and where it can be practically applied.

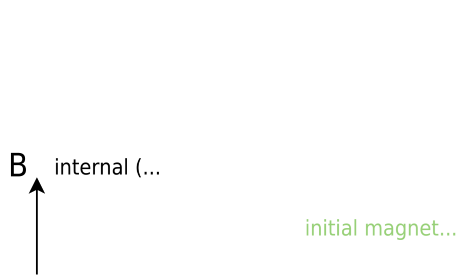
5.1 Linear Magnetic Circuits

For the upcoming calculations, the following assumptions are made

1. The relationship between B and H is linear: $B = \mu \cdot H$
   This is a good estimation when the magnetic field strength lays well below saturation
2. There is no stray field leaking out of the magnetic field conducting material.
3. The fields inside of airgaps are homogeneous. This is true for small air gaps.

One can calculate a lot of simple magnetic circuits when these assumptions and focusing on the average field line are applied.

Fig. 2: Simplifications and Linearization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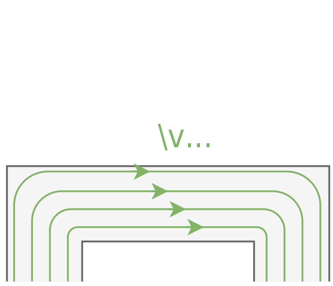
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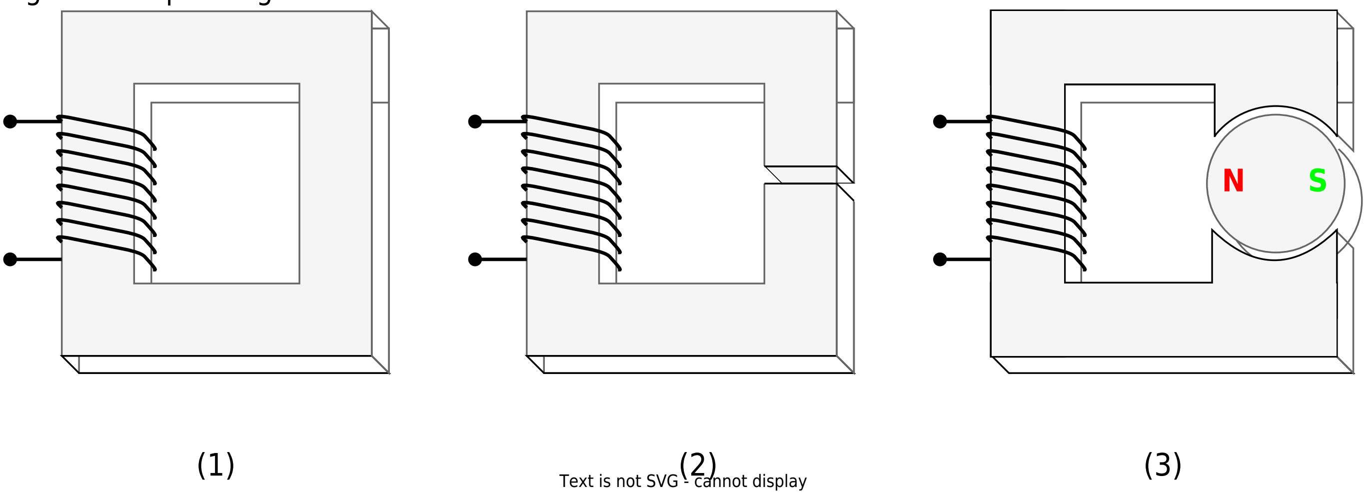
Two simple magnetic circuits are shown in figure 3: They consist of

- a current-carrying coil
- a ferrite core
- an airgap (in picture (2)).

Fig. 3: A simple magnetic Circuit



These three parts will be investigated shortly:

Current-carrying Coil

For the magnetic circuit, the coil is parameterized only by:

- its number of windings N and
- the passing current i.

These parameters lead to the magnetic voltage $\theta = N\cdot i$.

Ferrite Core

- The core is assumed to be made of ferromagnetic material.
- Therefore, the relative permeability in the core is much larger than in air ($\mu_{\rm r,core}\gg 1$).
- The ferrite core is also filling the inside of the current-carrying coil.
- The ferrite core conducts the magnetic flux around the magnetic circuit (and by this: also to the airgap)

Airgap

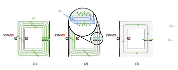
- The air gap interrupts the ferrite core.
- The width of the air gap is small compared to the dimensions of the cross-section of the ferrite core.
- The field in the air gap can be used to generate (mechanical) effects within the air gap.
  An example of this can be the force onto a permanent magnet (see figure 3 (3)).

With the above-mentioned assumptions the magnetic flux Φ must remain constant along the ferrite core, so $\Phi_{\rm core}=\rm const.$.
Since the magnetic field lines neither show sources nor sinks, also the flux passing over to the airgap

must be $\Phi_{\rm airgap}=\Phi_{\rm core}=\rm const.$ This can also be seen in figure 4 (1)).
A different view of this is the closed surface $\vec{A}$ (figure 4 (2)): Based on the examination in Recap of magnetic Field we know that the flux into the volume must be equal the flux out of the volume, or $\Phi_{\rm m} = \iint_{A} \vec{B} \cdot d \vec{A} = 0$.

The relationship $B=\mu \cdot H$, and $\mu_{\rm core}\gg\mu_{\rm airgap}$ lead to the fact that the ${H}$-Field must be much stronger within the airgap (figure 4 (3)).

Fig. 4: B- and H-field along the ferrite core



Magnetic Circuit in a Formula

Therefore, the following formula is given
$$\begin{align*} \Phi_{\rm m} &= &\iint_A &\vec{B} \cdot d \vec{A} &= &{\rm const.} \\ &= & & B \cdot A &= &{\rm const.} \\ &= & &B_{\rm core} \cdot A_{\rm core} &= & B_{\rm airgap} \cdot A_{\rm airgap} = {\rm const.} \\ \end{align*}$$

The assumption, that there is that the field inside of airgap is homogeneous and there is no stray field lead to the fact, that $A_{\rm core} = A_{\rm airgap}$.
Therefore:
$$\begin{align*} B_{\rm core} &= &B_{\rm airgap} = B \\ \mu_0 \mu_{\rm r,core} H_{\rm core} &= &\mu_0 \mu_{\rm r,airgap} H_{\rm airgap} = {{\Phi}\over{A}} \tag{5.2.1} \end{align*}$$

Besides this, the magnetic field strength H along one field line is directly given by:
$$\begin{align*} \theta = N \cdot i &= \int_s \vec{H} d\vec{s} = \\ &= \int_{\rm core} \vec{H} d\vec{s} + \int_{\rm airgap} \vec{H} d\vec{s} \\ \end{align*}$$

With the assumtion of a linear and homogeneous B-Field and the width δ of the airgap, this leads to:
$$\begin{align*} \theta = {H}_{\rm core} l_{\rm core} + {H}_{\rm airgap} \delta \tag{5.2.2} \end{align*}$$

With the prevoius formula $5.2.1$, this gets to:
$$\begin{align*} \theta &= {{B}\over{\mu_0 \mu_{\rm r,core}}} l_{\rm core} + {{B}\over{\mu_0 \mu_{\rm r,airgap}}} \delta \\ &= {{\Phi \cdot l_{\rm core}}\over{A \cdot \mu_0 \mu_{\rm r,core}}} + {{\Phi \cdot \delta}\over{A \cdot \mu_0 \mu_{\rm r,airgap}}} \\ &= {{1}\over{\mu_0 \mu_{\rm r,core}}}{{l_{\rm core}}\over{A}} \cdot \Phi + {{1}\over{\mu_0 \mu_{\rm r,airgap}}}{{\delta}\over{A}} \cdot \Phi \tag{5.2.3} \end{align*}$$

Comparing the formula $5.2.3$ with the ohmic resistance and resistivity of two resistors in series shows something interesting:
$$\begin{align*} u &= &R_1 \cdot &i &+ &R_2 \cdot i \\ &= &\rho {{l_1}\over{A_1}} \cdot &i &+ &\rho {{l_2}\over{A_2}} \cdot i \end{align*}$$

This leads to:

- The magnetic voltage θ acts like the electic voltage u,
  the magnetic flux Φ like the current i.
- The linear relationship $\theta = f(\Phi)$ is also called Hopkinson's Law.

Notice:

- Also for the magnetic circuit one can set up a lumped circuit model (see figure 5).
- Similar to Ohms law, there is a **magnetic resistance** or **reluctance**:
  \begin{align*} \boxed{R_{\rm m} = {{1}\over{\mu_0 \mu_{\rm r}}}{{l}\over{A}}} \end{align*}
- The unit of $R_{\rm m}$ is $[R_{\rm m}]= [\theta]/[\Phi] = ~\rm 1 A / Vs = 1/H $
- The length l is given by the mean magnetic path length (= average field line length in the core).
- Kirchhoff's laws (mesh rule and nodal rule) can also be applied:
  - The sum of the magentic fluxes Φ_i in into a node is: $\sum_i \Phi_i=0 $
  - The sum of the magnetic voltages θ_i along the average field line is: $\sum_i \theta_i=0 $
- The application of the lumped circuit model is based on multiple assumptions.
  In contrast to the simplification for the electric current and voltage the simplification for the flux and magnetic voltage is not as exact.

Fig. 5: lumped Circuit Model for magnetic Circuits

Application and Limitations of the Circuit Interpretation

Notice:

“Recipe” for solving magnetic circuits:

1. Watch out for individual magnetic resistances in the circuits:
    1. Separate the magnetic circuit into parts, where the permeability and area of the cross-section are constant.
       For these parts B and H is constant. These parts will have individual magnetic resistances.
    2. Each airgap also gets an individual magnetic resistance.
    3. Calculate the magnetic resistance by the mean magnetic path length through the individual parts.
2. Calculate the magnetic circuit as a circuit.
    1. Magnetic voltages depict voltage sources; magnetic flux depicts currents.
    2. Use the known way to solve the circuit.

Be aware that the orientation of the current I and the field strength $\vec{H}$ are also connected by the right-hand rule.

The results are only allowed as a first approximation. The following table shows the contrast between the electric and magnetic fields:

Property	**Electric Field**	**Magnetic Field**
Materials	There are “pure” isolators, which are completely non-conductive.	All materials have a permeability >0
Sources	The source is concentrated (= there are field sources / electric charges)	The magnetic source (= coil) is distributed
Simplifications	The simplifications often work for good results (small wire diameter, relatively constant resistivity)	The simplification is often too simple (widespread beyond the mean magnetic path length, non-linearity of the permeability)

Task 5.1.1 Coil on a plastic Core

A coil is set up onto a toroidal plastic ring ($\mu_{\rm r}=1$) with an average circumference of $l_R = 300 ~\rm mm$. The $N=400$ windings are evenly distributed along the circumference. The diameter on the cross-section of the plastic ring is $d = 10 ~\rm mm$. In the windings, a current of $I=500 ~\rm mA$ is flowing.

Calculate

1. the magnetic field strength H in the middle of the ring cross-section.
2. the magnetic flux density B in the middle of the ring cross-section.
3. the magnetic resistance $R_{\rm m}$ of the plastic ring.
4. the magnetic flux Φ.

Result

1. $H = 667 ~\rm {{A}\over{m}}$
2. $B = 0.84 ~\rm mT$
3. $R_m = 3 \cdot 10^9 ~\rm {{1}\over{H}}$
4. $\Phi = 66 ~\rm nVs$

Task 5.1.2 magnetic Resistance of a cylindrical coil

Calculate the magnetic resistances of cylindrical coreless (=ironless) coils with the following dimensions:

Result

1. $l=35.8~\rm cm$, $d=1.90~\rm cm$
2. $l=11.1~\rm cm$, $d=1.50~\rm cm$

Solution

1. $1.00\cdot 10^9 ~\rm {{1}\over{H}}$
2. $0.50\cdot 10^9 ~\rm {{1}\over{H}}$

The magnetic resistance is given by: \begin{align*} \ R_{\rm m} &= {{1}\over{\mu_0 \mu_{\rm r}}}{{l}\over{A}} \end{align*}

With

- the area $ A = \left({{d}\over{2}}\right)^2 \cdot \pi $
- the vacuum magnetic permeability $\mu_{0}=4\pi\cdot 10^{-7} ~\rm H/m$, and
- the relative permeability $\mu_{\rm r}=1$.

Task 5.1.3 magnetic Resistance of an airgap

Calculate the magnetic resistances of an airgap with the following dimensions:

1. $\delta=0.5~\rm mm$, $A=10.2~\rm cm^2$
2. $\delta=3.0~\rm mm$, $A=11.9~\rm cm^2$

Result

1. $3.9\cdot 10^5 ~\rm {{1}\over{H}}$
2. $2.0\cdot 10^6 ~\rm {{1}\over{H}}$

Task 5.1.4 Magnetic Voltage

Calculate the magnetic voltage necessary to create a flux of $\Phi=0.5 ~\rm mVs$ in an airgap with the following dimensions:

1. $\delta=1.7~\rm mm$, $A=4.5~\rm cm^2$
2. $\delta=5.0~\rm mm$, $A=7.1~\rm cm^2$

Result

1. $\theta = 1.5\cdot 10^3 ~\rm A$, or 1000 windings with $1.5~\rm A$
2. $\theta = 2.8\cdot 10^3 ~\rm A$, or 1000 windings with $2.8~\rm A$

Task 5.1.5 Magnetic Flux

Calculate the magnetic flux created on a magnetic resistance of $R_m = 2.5 \cdot 10^6 ~\rm {{1}\over{H}}$ with the following magnetic voltages:

1. $\theta = 35 ~\rm A$
2. $\theta = 950 ~\rm A$
3. $\theta = 2750 ~\rm A$

Result

1. $\Phi =14 ~\rm µVs$
2. $\Phi =0.38~\rm mVs$
3. $\Phi =1.1 ~\rm mVs$

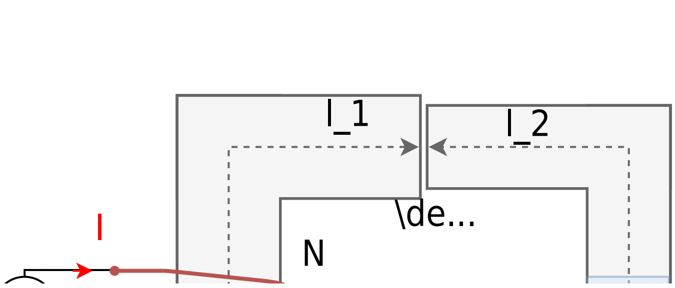
Task 5.1.6 Two-parted ferrite Core

A core shall consist of two parts as seen in figure 6. In the coil, with 600 windings shall pass the current $I=1.30 ~\rm A$.

The cross sections are $A_1=530 ~\rm mm^2$ and $A_2=460 ~\rm mm^2$. The mean magnetic path lengths are $l_1 = 200 ~\rm mm$ and $l_2 = 130 ~\rm mm$.

The air gaps on the coupling joint between both parts have the length $\delta=0.23 ~\rm mm$ each. The permeability of the ferrite is $\mu_r = 3000$. The cross-section area A_{δ} of the airgap can be considered the same as A_2

Fig. 6: Two-parted ferrite Core



1. Draw the lumped circuit of the magnetic system
2. Calculate all magnetic resistances $R_{\rm m,i}$
3. Calculate the flux in the circuit

Result

1. -
2. magnetic resistances: $R_{\rm m,1} = 100 \cdot 10^3 ~\rm {{1}\over{H}}$, $R_{\rm m,1} = 75 \cdot 10^3 ~\rm {{1}\over{H}}$, $R_{{\rm m},\delta} = 400 \cdot 10^3 ~\rm {{1}\over{H}}$
3. magnetic flux: $\Phi = 0.80 ~\rm mVs$

Task 5.1.7 Comparison with simplified Calculation

The magnetic circuit in figure 7 passes a magnetic flux density of $0.4 ~\rm T$ given by an excitation current of $0.50 ~\rm A$ in 400 windings. At position $\rm A-B$, an air gap will be inserted. After this, the same flux density will be reached with $3.70 ~\rm A$

Fig. 7: Example of a magnetic circuit

1. Calculate the length of the airgap δ with the simplification $\mu_{\rm r} \gg 1$
2. Calculate the length of the airgap δ exactly with $\mu_{\rm r} = 1000$

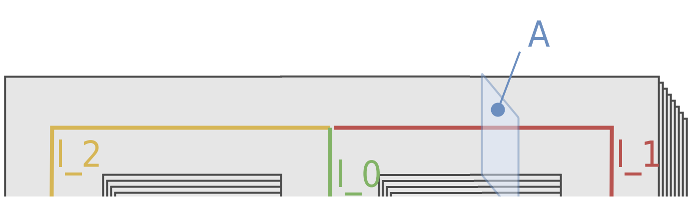
Result

1. $\delta = 4.02(12) ~\rm mm$
2. $\delta = 4.02(52) ~\rm mm$

Task 5.1.8 Coil on a ferrite Core with airgap

The choke coil shown in figure 8 shall be given, with a constant cross-section in all legs l_0, l_1, l_2. The number of windings shall be N and the current through a single winding I.

Fig. 8: Example for a Choke Coil



1. Draw the lumped circuit of the magnetic system
2. Calculate all magnetic resistances $R_{{\rm m},i}$
3. Calculate the partial fluxes in all the legs of the circuit

5.2 Non-linear magnetic Circuits

not included in the present script

5.3 Mutual Induction and Coupling

Situation: Two coils 1 and 2 near each other.
Questions:

- Which effect do the coils have on each other?
- Can we describe the effects with mainly electric properties (i.e. no geometric properties)

Fig. 9: Mutual Induction of two Coils

Effect of Coils on each other

1. Windings N_1 of coil 1 gets feed with current i_1.
2. Coil 1 generates change in flux Φ_{11}
3. Coil 2 gets passed by part of the flux (Φ_{21})
4. Stray flux $\Phi_{\rm S1}$ takes not part in coupling

- $\rightarrow$ Coils are magnetically coupled:
  Flux Φ_{11} of coil 1 gets divided into flux Φ_{21} in coil 2 and stray flux $\Phi_{\rm S1}$ not passing coil 2:

  $\Phi_{11} = \Phi_{21} + \Phi_{\rm S1}$
- Induced voltage in coil 2:

  $ u_{\rm ind,2} = -{{\rm d}\over{{\rm d}t}}\Psi_{21} $
  $\quad\quad = -N_2 \cdot {{\rm d}\over{{\rm d}t}}\Phi_{21} $
- Similar effect on coil 1 due to a current i_2 through coil 2:

  $\Phi_{22} = \Phi_{12} + \Phi_{\rm S2}$

$$u_{\rm ind,1} = -N_1 \cdot {{\rm d}\over{{\rm d}t}}\Phi_{12}$$

Linked Fluxes

For the single coil, we got the relationship between the linked flux Ψ and the current i as: $\Psi = L \cdot i$.
Now the coils also are interacting with each other. This must also be reflected in the relationship $\Psi_1 = f(i_1, i_2)$, $\Psi_2 = f(i_1, i_2)$:

$$\begin{align*} \Psi_1 &= &\Psi_{11} &+ \Psi_{12} \\ \Psi_2 &= &\Psi_{22} &+ \Psi_{21} \\ \\ \Psi_1 &= &L_{11} \cdot i_1 &+ M_{12} \cdot i_2 \\ \Psi_2 &= &L_{22} \cdot i_2 &+ M_{21} \cdot i_1 \\ \end{align*}$$

With

- L_{11} and L_{22} as the self induction
- M_{12} and M_{21} as the **mutual induction**

The formula can also be described as:

$$\begin{align*} \left(\begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) = \left(\begin{array}{c} L_{11} & M_{12} \\ M_{21} & L_{22} \end{array} \right) \cdot \left(\begin{array}{c} i_1 \\ i_2 \end{array} \right) \end{align*}$$

The view of the magnetic flux is sometimes good when effects like an acting Lorentz force in of interest. More often the coils are coupling two electric circuits linked in a transformer or a wireless charger. Here, the effect on the circuits is of interest. This can be calculated with the induced electric voltages $u_{\rm ind,1}$ and $u_{\rm ind,2}$ in each circuit. They are given by the formula $u_{{\rm ind},x} = -{\rm d}\Psi_x /{\rm d}t$:

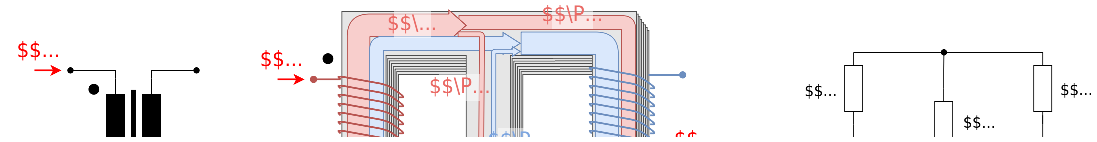
$$\begin{align*} \left(\begin{array}{c} u_1 \\ u_2 \end{array} \right) &= -{{\rm d}\over{{\rm d}t}} \left(\begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) \\ &= -\left(\begin{array}{c} L_{11} & M_{12} \\ M_{21} & L_{22} \end{array} \right) \cdot \left(\begin{array}{c} {{\rm d}\over{{\rm d}t}} i_1 \\ {{\rm d}\over{{\rm d}t}}i_2 \end{array} \right) \end{align*}$$

The main question is now: How do we get L_{11}, M_{12}, L_{22}, M_{21}?

Magnetic Circuit with two Sources

To get the self-induction and mutual induction of two interacting coils, we are going to investigate two coils on an iron core with a middle leg (see figure 10).
There the stray flux of the previous situation is only located in the middle leg. This also means, that there is no stray flux outside of the iron core.

Fig. 10: Example for Iron Core with two Coils



The figure 10 shows the fluxes on each part. The black dots nearby the windings mark the direction of the windings, and therefore the sign of the generated flux.
All the fluxes caused by currents flowing into the marked pins are summed up positively in the core. When there is a current flowing into a non-marked pin, its flux has to be subtracted from the others.

To get L_{11} and L_{22}, we look back to the inductance L of a long coil with the length l. This was given in the chapter Self-Induction as

\begin{align*} L &= {{\Psi}\over{i}} \\ L &= \mu_0 \mu_{\rm r} \cdot N^2 \cdot {{A} \over {l}} \\ \end{align*}

In this case, Ψ was the flux through the coil generated from the coil itself - here L_{11} and L_{22}.
Comparing this formula with the magnetic resistance $R_{\rm m} = {{1}\over{\mu_0 \mu_{\rm r}}} {{l}\over{A}}$ of the coil, one can conclude:

\begin{align} L = {{N^2 }\over {R_{\rm m,coil}}} \end{align}

Important here is, that we assumed no stray field. When taking this into account for calculating the self-induction the magnetic resistance $R_{\rm m,coil}$ ends op to be the resistance of the magnetic circuit, seen by the magnetic voltage source!

\begin{align} \boxed{L = {{N^2 }\over {R_{\rm m1}}}} \end{align}

The magnetic resistance $R_{\rm m1}$ in the example figure 10 is $R_{\rm m1} = R_{\rm m,11} + (R_{\rm m,ss}||R_{\rm m,22})$. Based on this formula, the self-inductions L_{11} and L_{22} can be written with the magnetic resistances of the coils as:

\begin{align*} L_{11} &= {{N_1^2 }\over {R_{\rm m1}}} \\ L_{22} &= {{N_2^2 }\over {R_{\rm m2}}} \end{align*}

To get the effect of the mutual induction, a coupling coefficient k is introduced. k_{21} describes how much of the flux from coil 1 is acting on coil 2 (similar for k_{12}):

$$\begin{align*} k_{21} = {{\Phi_{21} \cdot \Phi_{22} }\over{|\Phi_{21} \cdot \Phi_{22}|}} {{\Phi_{21}}\over{\Phi_{11}}} \\ \end{align*}$$

When $k_{21}=100~\%$, there is no flux in the middle leg but only in the second coil.
For $k_{21}=0~\%$ all the flux is in the middle leg circumventing the second coil, i.e. there is no coupling.

The mutual induction M_{21} can be calculated as the fraction of the linked flux Ψ_{11} in coil 2 based on the current i_1 from the coil 1 :

$$\begin{align*} M_{21} &= {{\Psi_{21}} \over{i_1}} \\ &= {{N_2 \cdot \Phi_{21}}\over{i_1}} \\ &= {{N_2 \cdot k_{21} \cdot \Phi_{11}}\over{i_1}} \\ &= {{N_2}\over{N_1}}k_{21} \cdot {{\Psi_{11}}\over{i_1}} \\ &= {{N_2}\over{N_1}}k_{21} \cdot L_{11} \\ &= {{N_2}\over{N_1}}k_{21} \cdot {{N_1^2 } \over {R_{\rm m1}}} \\ &= k_{21} \cdot {{N_1 \cdot N_2 }\over {R_{\rm m1}}} \\ \end{align*}$$

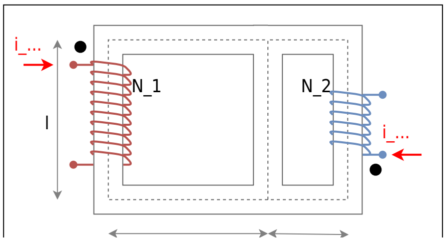
The formula is finally: $$\begin{align*} \left(\begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) = \left(\begin{array}{c} {{N_1^2}\over{R_{\rm m1}}} & k_{12}\cdot{{N_1 \cdot N_2}\over{R_{\rm m2}}} \\ k_{21}\cdot{{N_1 \cdot N_2}\over{R_{\rm m1}}} & {{N_2^2}\over{R_{\rm m2}}} \end{array} \right) \cdot \left(\begin{array}{c} i_1 \\ i_2 \end{array} \right) \end{align*}$$

Task 5.3.1 Example for magnetic Circuit with two Sources

The magnetical configuration in figure 11 shall be given.
The area of the cross-section is $A=9 ~\rm cm^2$ in all parts, the permeability is $\mu_r=800$, the length $l=12 ~\rm cm$ and the number of windings $N_1 = 400$, $N_2=300$. The coupling factors are $k_{12}=0.6$ and $k_{21}=0.8$.

Calculate L_{11}, M_{12}, L_{22}, M_{21}.

Fig. 11: Example for Iron Core with two Coils



Step 1: Draw the problem as a network

Step 2: Calculate the magnetic resistances

The magnetic resistance is summed up by looking at the circuit from the source 1:
$$\begin{align*} R_{\rm m1} &= R_{\rm m,11} + R_{\rm m,ss} || R_{\rm m,22} \\ \end{align*}$$

where the parts are given as
$$\begin{align*} R_{\rm m,11} &= {{1}\over{\mu_0 \mu_{\rm r}}}{{3\cdot l}\over{A}} \\ R_{\rm m,ss} &= {{1}\over{\mu_0 \mu_{\rm r}}}{{1\cdot l}\over{A}} \\ R_{\rm m,22} &= {{1}\over{\mu_0 \mu_{\rm r}}}{{2\cdot l}\over{A}} \\ \end{align*}$$

With the given geometry this leads to
$$\begin{align*} R_{\rm m1} &= {{1}\over{\mu_0 \mu_{\rm r}}}{{l}\over{A}}\cdot (3 + {{1\cdot 2}\over{1 + 2}}) \\ &= {{1}\over{\mu_0 \mu_{\rm r}}}{{l}\over{A}}\cdot {{11}\over{3}} \\ &= 133 \cdot 10^{3} \cdot {{11}\over{3}} ~\rm {{1}\over{H}}\\ \end{align*}$$

Similarly, the magnetic resistance R_{m2} is
$$\begin{align*} R_{\rm m2} &= {{1}\over{\mu_0 \mu_{\rm r}}}{{l}\over{A}}\cdot {{11}\over{4}} \\ &= 133 \cdot 10^{3} \cdot {{11}\over{4}} ~\rm {{1}\over{H}}\\ \end{align*}$$

Step 3: Calculate the magnetic inductances

\begin{align*} L_{11} &= {{N_1^2}\over{R_{\rm m1}}} &= 329 ~\rm mH\\ \\ L_{22} &=

{{N_2^2}\over{R_{\rm m2}}} &= 247 ~\rm mH\\ \\ M_{21} &= k_{21}\cdot{{N_1 \cdot N_2}\over{R_{\rm m1}}} &= 197 ~\rm mH\\ \\ M_{12} &= k_{12}\cdot{{N_1 \cdot N_2}\over{R_{\rm m2}}} &= 197 ~\rm mH\\ \end{align*}

For symmetrical magnetic structures and $\mu_{\rm r} = \rm const.$ the following applies:

- the mutual inductances are equal: $M_{12} = M_{21} = M$
- the mutual inductance M is: $M = \sqrt{M_{12}\cdot M_{21}} = k \cdot \sqrt {L_{11}\cdot L_{22}}$
- The resulting *total coupling* k is given as \begin{align*} k = \sqrt{k_{12}\cdot k_{21}} \end{align*}

Effects in the electric Circuits

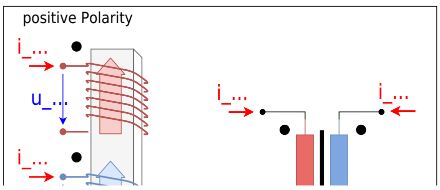
- Whenever two coils are magnetically coupled, not only the self-induction L but also the mutual induction M applies.
- Based on the currents i_1, i_2 in the two circuits, the induced voltages are given by:

\begin{align*} u_1 &= L_{11} \cdot {{{\rm d}i_1}\over{{\rm d}t}} &+ M \cdot {{{\rm d}i_2}\over{{\rm d}t}} & \\ u_2 &= M \cdot {{{\rm d}i_1}\over{{\rm d}t}} &+ L_{22} \cdot {{{\rm d}i_2}\over{{\rm d}t}} & \\ \end{align*}

It is important to consider the polarity of the fluxes for the calculation in circuits (see figure 15). The **sign of the mutual induction** is influenced by

- the direction of the windings, and
- the orientation/counting of the current in the circuit.

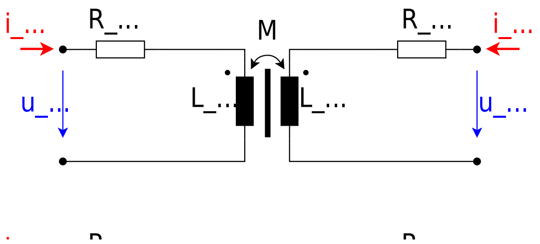
Fig. 15: Polarity of Coupling



positive Polarity

The polarity is positive when both currents either flow into or out of the dotted pin (see figure 16).

Fig. 16: Example Circuits with positive Polarity



In this case, the **mutual induction added positively**.

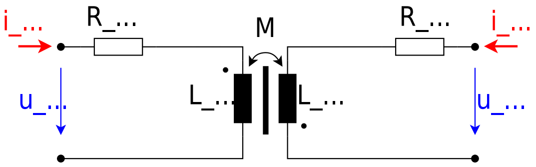
The formula of the shown circuitry is then:

$$\begin{align*} u_1 &= R_1 \cdot i_1 &+ L_{11} \cdot {{{\rm d}i_1}\over{{\rm d}t}} &+ M \cdot {{{\rm d}i_2}\over{{\rm d}t}} & \\ u_2 &= R_2 \cdot i_2 &+ L_{22} \cdot {{{\rm d}i_2}\over{{\rm d}t}} &+ M \cdot {{{\rm d}i_1}\over{{\rm d}t}} & \\ \end{align*}$$

negative Polarity

The polarity is negative when only one current either flows into the dotted pin and the other one out of the dotted pin (see figure 17).

Fig. 17: Example Circuits with negative Polarity



In this case, the **mutual induction added negatively**.

The formula of the shown circuitry is then: $$\begin{align*} u_1 &= R_1 \cdot i_1 &+ L_{11} \cdot {{{\rm d}i_1}\over{{\rm d}t}} &- M \cdot {{{\rm d}i_2}\over{{\rm d}t}} & \\ u_2 &= R_2 \cdot i_2 &+ L_{22} \cdot {{{\rm d}i_2}\over{{\rm d}t}} &- M \cdot {{{\rm d}i_1}\over{{\rm d}t}} & \\ \end{align*}$$

Task 5.3.2 toroidal Core with two Coils

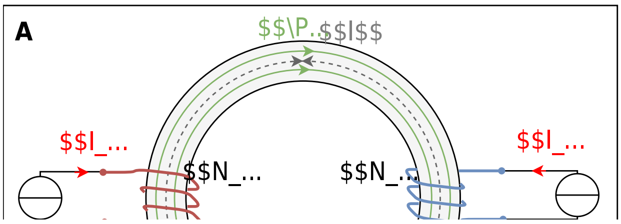
A toroidal core (ferrite, $\mu_{\rm r} = 900$) has a cross-sectional area of $A = 500 ~\rm mm^2$ and an average circumference of $l=280 ~\rm mm$. On the core, there are two coils $N_1=500$ and $N_2=250$ wound. The currents on the coils are $I_1 = 250 ~\rm mA$ and $I_2=300 ~\rm mA$.

Result

1. $0.10 ~\rm mVs$
2. $0.40 ~\rm mVs$

1. The coils shall pass the currents with positive polarity (see the image **A** in figure 18). What is the resulting magnetic flux $\Phi_{\rm A}$ in the coil?
2. The coils shall pass the currents with negative polarity (see the image **B** in figure 18). What is the resulting magnetic flux $\Phi_{\rm B}$ in the coil?

Fig. 18: toroidal core with two coils in positive and negative polarity



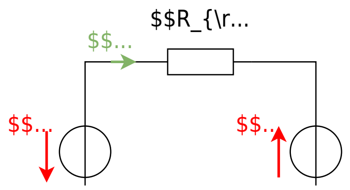
Solution

The resulting flux can be derived from a superposition of the individual fluxes $\Phi_1(I_1)$ and $\Phi_2(I_2)$, or alternatively by summing the magnetic voltages in the loop ($\sum_x \theta_x = 0$).

Step 1 - Draw an equivalent magnetic circuit

Since there are no branches all of the core can be lumped to a single magnetic resistance (see figure 19).

Fig. 19: equivalent magnetic circuit



Step 2 - Get the absolute values of the individual fluxes

Hopkinson's Law can be used here as a starting point.
It connects the magnetic flux Φ and the magnetic voltage θ on the single magnetic resistor R_m.
It also connects the single magnetic fluxes Φ_x (with $x = {1,2}$) and the single magnetic voltages θ_x.

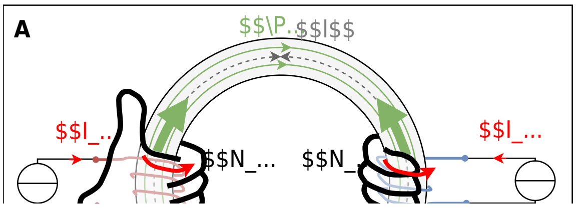
\begin{align*} \theta_x &= R_{\rm m} \cdot \Phi_x \\ N_x \cdot I_x &= {{1}\over{\mu_0 \mu_{\rm r}}}{{l}\over{A}} \cdot \Phi_x \\ \rightarrow \Phi_x &= \mu_0 \mu_{\rm r} {{A}\over{l}} \cdot N_x \cdot I_x = {{1}\over{R_{\rm m} }} \cdot N_x \cdot I_x \\ \end{align*}

With the given values we get: $R_{\rm m} = 495 {\rm {kA}\over{Vs}}$

Step 3 - Get the signs/directions of the fluxes

The figure 20 shows how to get the correct direction for every single flux by use of the right-hand rule.
The fluxes have to be added regarding these directions and the given direction of the flux in question.

Fig. 20: toroidal core with two coils in positive and negative polarity



Therefore, the formulas are

$$\begin{align*} \Phi_{\rm A} &= \Phi_{1} - \Phi_{2} \\ &={{1}\over{R_{\rm m} }} \cdot \left(N_1 \cdot I_1 - N_2 \cdot I_2 \right) \\ & = 0.25 ~{\rm mVs} - 0.15 ~{\rm mVs} \\ \Phi_{\rm B} &= \Phi_{1} + \Phi_{2} \\ &={{1}\over{R_{\rm m} }} \cdot \left(N_1 \cdot I_1 + N_2 \cdot I_2 \right) \\ & = 0.25 ~{\rm mVs} + 0.15 ~{\rm mVs} \end{align*}$$

5.4 Magnetic Energy

The magnetic field of a coil stores magnetic energy. The energy transfer from the electric circuit to the magnetic field is also the cause of the “current dampening” effect of the inductor. The energetic turnover for charging an conductor from $i(t_0=0)=0$ to $i(t_1)=I$ is given by:

$$\begin{align*} W_m = \int_0^\infty u(t)\cdot i(t) {\rm d}t \end{align*}$$

With $u_L = L \cdot {\rm d}i/{\rm d}t$, this becomes:

$$\begin{align*} W_{\rm m} &= \int_0^\infty L\cdot {{{\rm d}i}\over{{\rm d}t}}\cdot i(t) {\rm d}t \\ &= \int_0^\infty L\cdot i(t) {\rm d}i \\ &= L\cdot \int_0^\infty i(t) {\rm d}i \\ &= L\cdot \left[{{1}\over{2}}i^2\right]_0^I \\ \boxed{W_m = {{1}\over{2}}L\cdot I^2 } \end{align*}$$

magnetic Energy of a magnetic Circuit

With this formula also the stored energy in a magnetic circuit can be calculated. For this, the formula be rewritten by the properties linked flux $\Psi = N \cdot \Phi = L \cdot I$ and magnetic voltage $\theta=N \cdot I = \Phi \cdot R_{\rm m}$ of the magnetic circuit:

$$\begin{align*} \boxed{W_{\rm m} = {{1}\over{2}} \Psi \cdot I = {{1}\over{2}} {{\Psi^2}\over{L}}= {{1}\over{2}}{{\Phi^2 }\over{N^2 \cdot L}} = {{1}\over{2}} \Phi^2 \cdot R_{\rm m} = {{1}\over{2}}{{\theta^2 }\over{R_{\rm m}}}} \end{align*}$$

magnetic Energy of a toroid Coil

The formula can also be used for calculating the stored energy of a toroid coil with N windings, the cross-section A, and an average length l of a field line. By this, the following formulas can be used:

$$\begin{align*} \theta = H \cdot l = N \cdot I \\ \Phi = B \cdot A \end{align*}$$

With the above-mentioned formulas of the magnetic circuit, we get:

$$\begin{align*} W_{\rm m} &= {{1}\over{2}} \Psi \cdot I \\ &= {{1}\over{2}}N \cdot B \cdot A \cdot {{H \cdot l}\over{N}} \\ &= {{1}\over{2}}B \cdot H \cdot A \cdot l \\ \end{align*}$$

$$\begin{align*} \boxed{W_{\rm m} = {{1}\over{2}}B\cdot H \cdot V} \end{align*}$$

The **magnetic energy density** $w_{\rm m}$ can therefore be calculated as:

$$\begin{align*} w_{\rm m} &= {{W_{\rm m}}\over{V}} \\ &= {{1}\over{2}} B \cdot H \\ \end{align*}$$

This formula is also true for other types of coils.

generalized magnetic Energy

The general term to find the magnetic energy (e.g. for inhomogeneous magnetic fields) is given by

$$\begin{align*} W_{\rm m} &= \iiint_V{w_{\rm m} {\rm d}V} \\ &= \iiint_V{\vec{B}\cdot \vec{H} {\rm d}V} \end{align*}$$

Application of the magnetic Energy

The circuit shown in figure 21 shall now be investigated. The inductor shall be a toroid coil with N windings, the cross-section A, and an average length l of a field line.

Fig. 21: Example Circuits for calculating the magnetic Energy

The Kirchhoff mesh law leads to:

\begin{align*} u_{\rm s} = u_R + u_L \\ u_{\rm s} = R \cdot i + N {{{\rm d}\Phi}\over{{\rm d}t}} \end{align*}

Multiplying with i and with dt we get the principle of conservation of energy $dw = u \cdot i \cdot {\rm d}t$ for each small time step.

\begin{align*} u_{\rm s} \cdot i \cdot dt &= R \cdot i^2 \cdot {\rm d}t & + N {{{\rm d}\Phi}\over{dt}} \cdot i \cdot {\rm d}t & \\ dW &= {\rm d}W_R & + {\rm d}W_m & \end{align*}

In this way, we get the magnetic energy as: \begin{align*} dW_{\rm m} &= N {{{\rm d}\Phi}\over{{\rm d}t}} \cdot i \cdot {\rm d}t \\ W_{\rm d} &= \int {\rm d}W_{\rm m} \\ &= N \int_0^t {{{\rm d}\Phi}\over{{\rm d}t}} \cdot i \cdot {\rm d}t \\ &= N \int_0^ {\Phi} i \cdot {\rm d}\Phi \\ \end{align*}

In a toroid coil with a given cross-section A the flux change ${\rm d}\Phi$ can only be given as a change in the field B. Therefore, ${\rm d}\Phi = A \cdot {\rm d}B$. Additionally, we know that the magnetic voltage is given by $\theta(t) = N \cdot i = H(t) \cdot l$. Including this in the formula gives us:

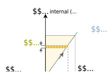
\begin{align*} W_{\rm m} &= N \int_0^{B} I \cdot A \cdot {\rm d}B \\ &= \int_0^{B} H(B) \cdot l \cdot A \cdot {\rm d}B \\ &= V \int_0^{B} H(B) \cdot {\rm d}B \\ \end{align*}

We can conclude that the magnetic energy $W_{\rm m}$: $W_{\rm m}$ can be calculated from the H-B curve by integrating the external magnetic field strength H for each small step of the flux density ${\rm d}B$. This will be shown for the case of a linear magnetic behavior, a nonlinear behavior, and the situation with magnetic hysteresis shortly.

Circuit with linear magnetic Behavior

In figure 22 the situation for a magnetic material with a linear relationship between B and H is shown. Given by the maximum current $I_{\rm max}$ the maximum field strength $H_{\rm max}$ can be derived. In the circuit in figure 21, the inductor will experience increasing and decreasing current. Therefore, the B-H-curve gets passed through positive and negative values of H and H along the line of $B=\mu H$.

Fig. 22: H-B-Curve for linear material



The situation for integrating the area in the graph is also shown: For each step ${\rm d}B$ the corresponding value of the field strength H has to be integrated. For $B_0=0$ to $B=B_{\rm max}$ the magnetic energy is

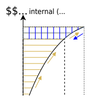
\begin{align*} W_{\rm m} &= V\int_0^{B} H(B) \cdot {\rm d}B \\ &= V\int_0^{B} {{B }\over{\mu}} \cdot {\rm d}B \\ &= {{1}\over{2}} V {{B^2}\over{\mu}} \\ &= {{1}\over{2}} V \cdot B \cdot H \\ \end{align*}

This situation is a good approximation for air or non-magnetic materials. However, it does not work well for ferrite materials, since they show nonlinear behavior and hysteresis.

Circuit with nonlinear magnetic Behavior

In figure 22 the situation for a magnetic material with a nonlinear relationship between B and H is shown.

Fig. 23: H-B-Curve for nonlinear material

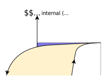


In this case, the permeability $\mu_{\rm r}$ is not a constant but can be represented as a function: $\mu_{\rm r}= f(B)$. Here, the formula $W_{\rm m} = V\int_0^{B} H(B) \cdot {\rm d}B$ also applies - so the magnetic energy is again the area between the curve and the B-axis. As an example, the situation of the field strength $H(t_1)=H_1$ is shown. This shall be the field strength after magnetizing the ferrite material to $H_{\rm max}$ (yellow arrows) and then partly demagnetizing the material again (blue arrow). The magnetization corresponds to an energy intake to the magnetic field and the demagnetization to an energy outtake.

Moving along the H-B-curve, one can see, that the energy intake and outtake are the same when coming back to a start point. This means that the magnetization and demagnetization take place lossless in this example. This is a good approximation for magnetically soft materials, however, does not work for magnetically hard materials like a permanent magnet. Here, hysteresis also has to be considered.

Circuit with magnetic Hysteresis

Fig. 24: H-B-Curve material with Hysteresis



Tasks

Task 5.1.9 Application: Shaded Pole Motor

The figure 25 and figure 25 show a shaded pole motor of a commercial oven.

- Find out how this motor works - explicitly: why is there a preferred direction of the motor?
- In which direction does the shown motor run?

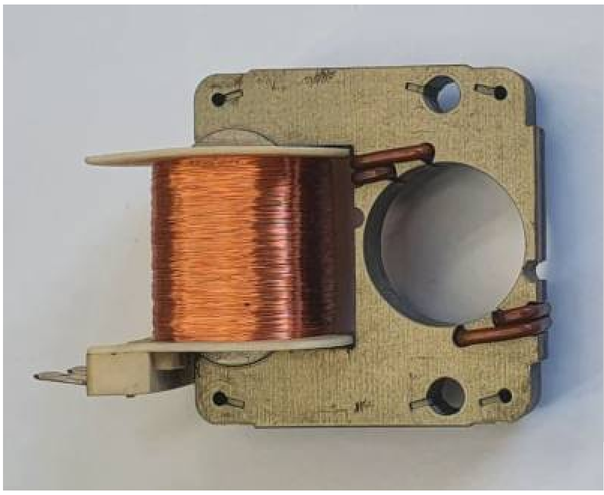


Fig. 25: Core of a shaded Pole Motor

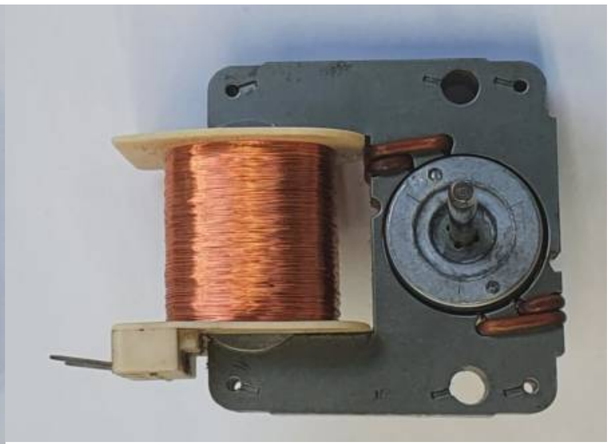


Fig. 26: Setup of the full shaded Pole Motor

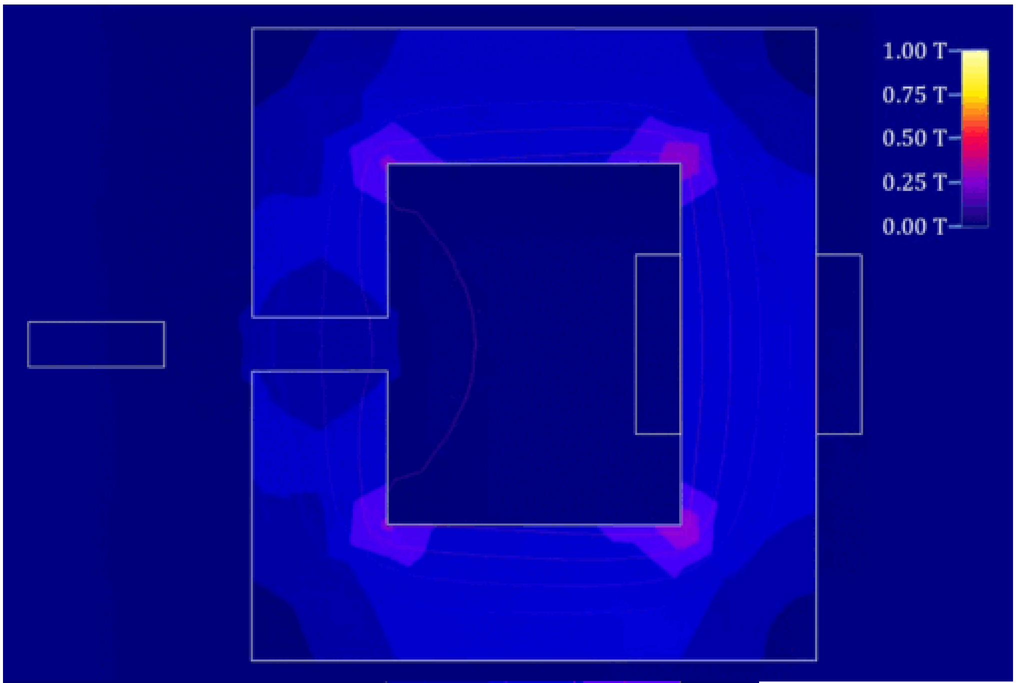
Task 5.1.10 Further exercise

The book DC Electrical Circuit Analysis - A Practical Approach (Fiore) has some nice exercise for beginning in the topic of magnetic circuits.

Further Information

An alternative interpretation of the magnetic circuits is the Gyrator–capacitor model. The big difference there is, that there the magnetic flux Φ is not interpreted as an analogy to the electric current I but to the electric charge Q. This model can solve more questions, however, is a bit less intuitive based on this course and less commonly used compared to the Magnetic_circuit, which was also presented in this chapter.

Moving a Plate into an Air Gap



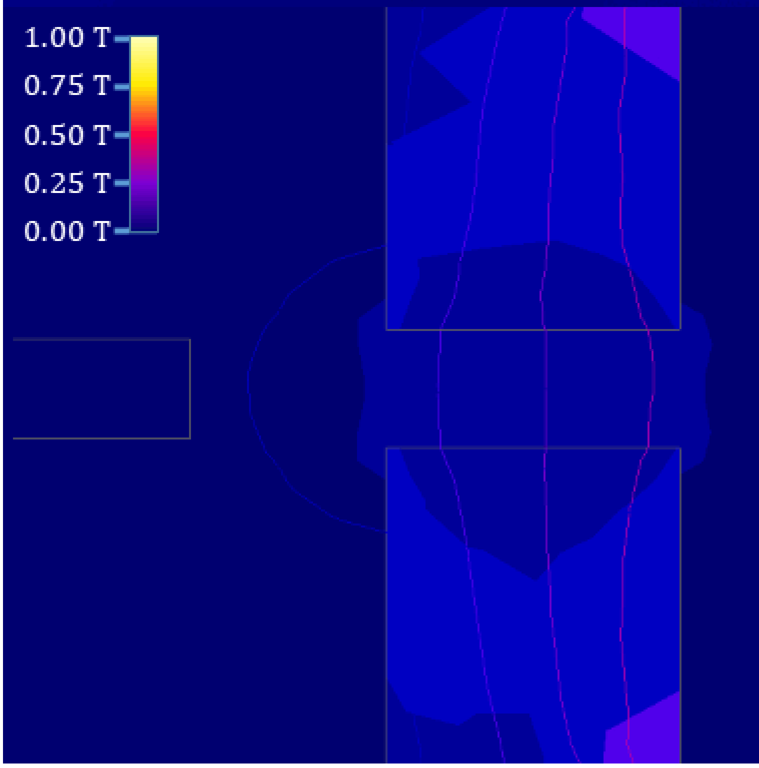


Fig. 27: a magnetic circuit with a moving plate

Switch Reluctance Motor

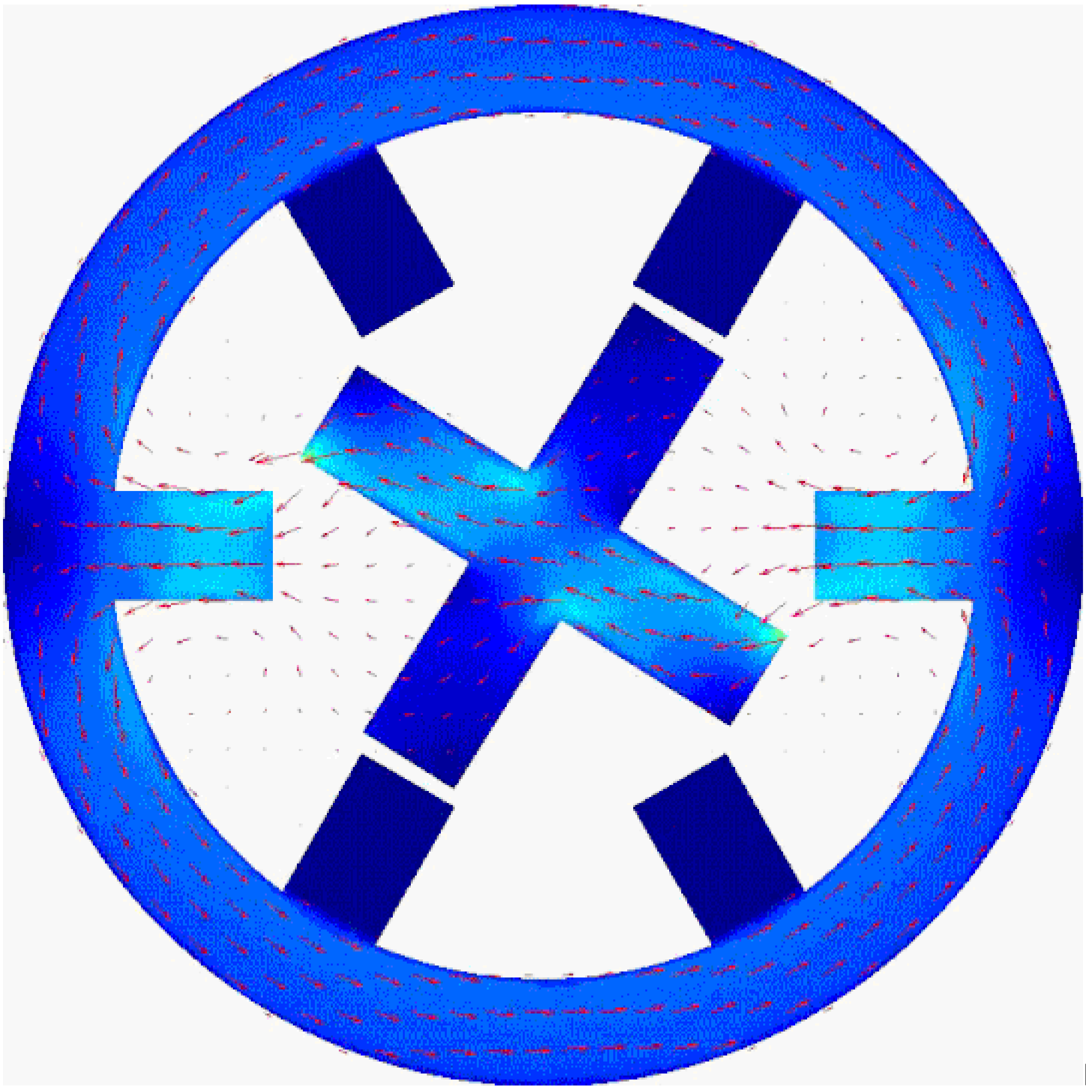


Fig. 28: a switch reluctance Motor as an application for the magnetic resistance

Based on a wikimedia image from Hamidreza D (CC-SA 4.0)

Resolver

Fig. 29: Resolver

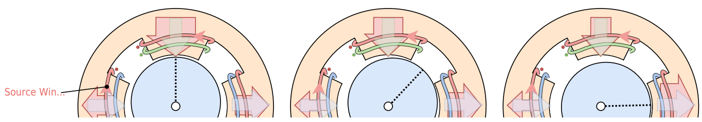


Fig. 30: Overview of the magnetic Formalism